COMPLEX ANALYSIS BACK PAPER EXAMINATION

Attempt all questions. Total Marks: 100. If you use a result proved in class then it is enough to just quote it.

(1) Let G be a domain. Identify all analytic functions $f : G \to \mathbb{C}$ such that $|f|$ is a constant function on G. (10 marks)

(2) Compute $\int_\gamma \frac{\log(z)}{z^n} dz$ where $\gamma(t) = 2 + e^{it}$ where $0 \leq t \leq 2\pi$ and $n \in \mathbb{N}$. (10 marks)

(3) Compute $\int_\gamma (\frac{z}{z-1})^n dz$ where $\gamma(t) = 1 + e^{it}$ for $0 \leq t \leq 4\pi$ and $n \in \mathbb{N}$. (10 marks)

(4) Let f be an entire function and suppose there exists a constant M, an $R > 0$ and an integer $n \geq 1$ such that $|f(z)| \leq M|z|^n$ for all $|z| > R$. Show that f is a polynomial of degree $\leq n$. (10 marks)

(5) Suppose $f : G \to \mathbb{C}$ is analytic and one-one. Show that $f'(z) \neq 0$ for all $z \in G$. (10 marks)

(6) Let G be a region, let $f_n : G \to \mathbb{C}$ be an analytic function for each $n \geq 1$. Suppose f_n converges uniformly to a function $f : G \to \mathbb{C}$. Show that f is an analytic function. (10 marks)

(7) Use the Residue Theorem to show that $\int_0^\infty \frac{1}{1+x^2} dx = \frac{\pi}{2}$. (10 marks)

(8) Find all possible values of $\int_\gamma \frac{1}{1+z^2} dz$ where γ is any closed, rectifiable curve in $\mathbb{C}$ not passing through $\pm i$. (10 marks)

(9) Prove that an entire function $f(z)$ has a pole of order m at infinity (that is, $f(z^{-1})$ has a pole of order m at 0), if and only if, f is a polynomial of degree m. (10 marks).

(10) How many zeroes does $f(z) = z^4 - 3z + 1$ have in the open unit disc? (10 marks)